MMAT5030 Notes 9

1. L^2-Theory

Recall some spaces:

- $R[-\pi, \pi]$ or $R_{2\pi}$, the space of Riemann integrable functions on $[-\pi, \pi]$,
- $R^1_{2\pi}$, the space of improperly Riemann integrable functions on $[-\pi, \pi]$,
- $R^2_{2\pi}$, the space of all functions whose squares are improperly Riemann integrable on $[-\pi, \pi]$,
- $R^1(\mathbb{R})$, the space of all improperly Riemann integrable functions on $(-\infty, \infty)$,
- $R^2(\mathbb{R})$, the space of all functions whose squares are improperly Riemann integrable on $(-\infty, \infty)$,
- $\mathcal{S}$, the Schwartz class consisting of all infinitely many times differentiable functions whose derivatives are all rapidly decreasing on $(-\infty, \infty)$.

It is known that neither $R^1(\mathbb{R})$ is contained in $R^2(\mathbb{R})$ nor the other way around. For instance, the function $f(x) = 1/\sqrt{x}, x \in (0, 1)$ and equals to 0 elsewhere belongs to $R^1(\mathbb{R})$ but not to $R^2(\mathbb{R})$ and the function $g(x) = 1/x, x \in (1, \infty)$, and equals to 0 elsewhere belongs to $R^2(\mathbb{R})$ but not to $R^1(\mathbb{R})$.

In Notes 4 we study L^2-theory in the periodic case. For $f \in R[-\pi, \pi]$, we proved that

$$\lim_{n\to\pm\infty} \|f - \sum_{k=-n}^{n} c_n e^{-nx}\| = 0 \ ,$$

as well as the Parseval's Identity

$$\|f\|^2 = 4\pi \sum_{n=-\infty}^{\infty} |c_n|^2 \ .$$

(It is

$$\|f\|^2 = \frac{\pi}{2} a_0^2 + \pi \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right),$$

where a_n and b_n are the Fourier coefficients of the real-valued function f.) We point that the same proofs actually show that these two results hold for all functions in $R^2_{2\pi}$.

Now, the Fourier transform maps $\mathcal{S}$ bijectively onto itself. We have

Proposition 9.1. For $f, g \in \mathcal{S}$,

(a)

$$\langle \widehat{f}, \widehat{g} \rangle = 2\pi \langle f, g \rangle \ ,$$

and

(b)

$$\|\widehat{f}\|^2 = 2\pi \|f\|^2 \ .$$

(a) shows that the Fourier transform preserves the inner product (up to a constant 2π) and (b) is the analogous Parseval's Identity.

Proof. We have

$$\begin{aligned}
2\pi\langle f, g\rangle &= 2\pi \int f(x)\overline{g(x)}\, dx \\
&= \int\int f(x)\overline{e^{i\xi x}\widehat{g}(\xi)}\, d\xi\, dx \qquad \text{(inverse Fourier transform)} \\
&= \int\int f(x)e^{-\xi x}\overline{\widehat{g}(\xi)}\, dx\, d\xi \\
&= \langle \widehat{f}, \widehat{g}\rangle \ .
\end{aligned}$$

Taking $f = g$, we obtain (b).

To describe the full L^2-theory we need to extend $R^2(\mathbb{R})$ to the Lebesgue space. Let $L^2(\mathbb{R})$ be the space consisting of all square Lebesgue integrable functions. We list some facts on this space:

- $R^2(\mathbb{R}) \subset L^2(\mathbb{R})$. In fact, the Lebesgue integral of a function is equal to its Riemann integrable whenever the latter exists.
- Every Cauchy sequence in $L^2(\mathbb{R})$ converges to some function in $L^2(\mathbb{R})$. This is the key property. $R^2(\mathbb{R})$ does not enjoy this property.
- Again $L^(\mathbb{R})$, which contains $R^1(\mathbb{R})$, does not belong to $L^2(\mathbb{R})$ and again $L^2(\mathbb{R})$ does not belong to $L^1(\mathbb{R})$. The same examples above confirm this.
- The Schwartz class $\mathcal{S} \subset L^2(\mathbb{R})$. Moreover, for each $f \in L^2(\mathbb{R})$, there is a sequence f_n in $\mathcal{S}$ satisfying $\|f - f_n\| \to 0$ as $n \to \infty$.

Whenever $\|f_n - f\| \to 0, f_n \in \mathcal{S}$ and $f \in L^2(\mathbb{R})$, applying the last property to the Parseval's Identity yields $\|\widehat{f_n} - \widehat{f_m}\|^2 = 2\pi\|f_n - f_m\|^2$ which implies that $\{\widehat{f_n}\}$ is a Cauchy sequence. Therefore, by the second fact it converges to some function $\varphi \in L^2(\mathbb{R})$. We define $\widehat{f} = \varphi$. It is not hard to see that this definition is independent of the choice of

$\{f_n\}$. It means that the Fourier transform can be extended from $\mathcal{S}$ to $\mathcal{S}$ to a linear map from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$. Using Proposition 9.1, it is easy to show

Theorem 9.2. The extension Fourier transform is a linear bijective map from $L^2(\mathbb{R})$ to itself. For $f, g \in L^2(\mathbb{R})$,

(a).

$$\langle \widehat{f}, \widehat{g} \rangle = 2\pi \langle f, g \rangle \ ,$$

and

(b).

$$\|\widehat{f}\|^2 = 2\pi \|f\|^2 \ .$$

2. Convolution of Functions

For f and g in $\mathcal{S}$, their **convolution** is defined to be

$$f * g(x) = \int f(x-y)g(y)\, dy \ .$$

Elementary properties of the convolution product are listed below.

Proposition 9.3. For $f, g, h \in \mathcal{S}$, then $f * g \in \mathcal{S}$ and the followings hold:

(a)

$$f * (ag + bh) = af * g + bf * h \ , \quad a, b \in \mathbb{C},$$

(b)

$$f * g = g * f \ ,$$

(c)

$$f * (g * h) = (f * g) * h \ ,$$

(d)

$$(f * g)' = f' * g = f * g' \ ,$$

(e)

$$\widehat{f * g} = \widehat{f}\widehat{g} \ ,$$

(f)

$$\widehat{f} * \widehat{g} = 2\pi \widehat{fg} \ .$$

The proofs of (a)-(f) are straightforward. The proof of (f) is done by showing that they are equal after taking Fourier transform which is done below. As Fourier transform is bijective on $\mathcal{S}$, (f) implies (e). Now, the left hand side is

$$\widehat{\widehat{f} * \widehat{g}}(x) = \widehat{\widehat{f}}(x)\widehat{\widehat{g}}(x) = 4\pi^2 f(-x)g(-x) ,$$

after using the formula $\widehat{\widehat{f}}(x) = 2\pi f(-x)$. On the other hand, the right hand side is

$$2\pi\widehat{\widehat{fg}}(x) = 4\pi^2 (fg)(-x) = 4\pi^2 f(-x)g(-x) ,$$

done.

We refer to 7.1 in Text for a discussion of the meaning of the convolution. It becomes relevant in Fourier transform for two reasons. First, (e) and (f) in Proposition 9.3 show that the ordinary pointwise product goes over to the convolution under the Fourier transform. Second, it is used in the proof of the inversion formula (which we omit, see Text for more).

3. Initial Value Problem for the Heat Equation

As an application of Fourier transform we derive a representation formula for the solution of the heat equation in the real line.

Consider the Cauchy problem for the 1-D heat equation

$$\begin{cases} u_t = \kappa u_{xx} \ , \quad \kappa > 0, & \text{in} \quad \mathbb{R} \times [0, T), \\ u(x, 0) = g(x) & x \in \mathbb{R}. \end{cases} \tag{1}$$

Cauchy problem is also called the initial value problem. There is no boundary condition when the underlying domain is the entire real number. To derive a formula for this Cauchy problem we take Fourier transform on both sides of the equation (fix t): By Proposition 8.2

$$\widehat{u_t}(\xi, t) = \kappa(i\xi)^2 \hat{u}(\xi, t) = -\kappa\xi^2 \widehat{u}(\xi, t) \ .$$

Regarding ξ as a parameter, this is a linear ODE whose solution is given solution is

$$\widehat{u}(\xi, t) = Ce^{-k\xi^2 t}.$$

To satisfy the initial condition, we should take $C = \widehat{g}(\xi)$. So taking inverse transform,

$$\begin{aligned} u(x, t) &= \frac{1}{2\pi} \int e^{ix\cdot\xi} \widehat{u}(\xi, t) d\xi \\ &= \frac{1}{2\pi} \int e^{ix\cdot\xi} e^{-\kappa\xi^2 t} \widehat{g}(\xi) d\xi \\ &= \frac{1}{2\pi} \int \int e^{ix\cdot\xi} e^{-\kappa\xi^2 t} e^{-i\xi y} g(y) dy d\xi \\ &= \frac{1}{2\pi} \int \left(\int e^{-\kappa\xi^2 t - i\xi(y-x)} d\xi \right) g(y) dy \ . \end{aligned}$$

From last lecture we have the formula

$$\mathcal{F}(e^{-ax^2/2}) = \sqrt{\frac{2\pi}{a}} e^{-\xi^2/2a} \ .$$

Taking $a = 2\kappa t$, we arrive at the following formula for the solution

$$u(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\kappa t}} g(y) dy \ . \tag{2}$$

It is a bit hard to justify all steps we get this formula, so the best way is to prove directly that this formula gives a solution to the initial value problem (1). In the following we simply take $\kappa = 1$. We define the **heat kernel** to be

$$K(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} .$$

Notice that K is not defined at $t = 0$. We have the following easily verified facts:

(a) For $t > 0$, K satisfies the heat equation for $\kappa = 1$,

(b) $K > 0$ and $\int K(z,t)dz = 1$, $\forall t > 0$.

(c) $\int_{|z| \geq \delta} K(z,t) \to 0$ as $t \to 0$, $\forall$ fixed $\delta > 0$.

Theorem 9.4 *Let u be the function defined (2) ($\kappa = 1$). Suppose that g is a continuous and bounded function in $\mathbb{R}$. Then u solves the heat equation for $(x,t) \in \mathbb{R} \times (0,\infty)$, and $u(x,t)$ tends to $g(x)$ as $t \to 0$.*

Proof. This proof can be skipped. For $t > 0$, the heat kernel decays very fast. Using this fact one can show differentiation and integration commute in the solution formula (2). Hence u solves the heat equation for positive time.

The real job is to show that u realizes the initial condition. To this end we define a function U in $\mathbb{R} \times [0,\infty)$ by setting

$$U(x,t) = \begin{cases} u(x,t), & x \in \mathbb{R}, \quad t > 0 \\ g(x), & x \in \mathbb{R}, \quad t = 0 \end{cases}$$

(Notice that the heat kernel is not well-defined at $t = 0$. So u is only defined for $t > 0$). We need to show that $U \in C(\mathbb{R} \times [0,\infty))$. Since U is continuous for $t > 0$, it suffices to consider its continuity at $(x_0, 0)$.

Fix $x_0 \in \mathbb{R}$ and for $t > 0$. By Fact (b)

$$\begin{aligned} & |U(x,t) - U(x_0,0)| \\ = \ & |u(x,t) - g(x_0)| \\ = \ & \left| \int K(x-y,t) g(y) dy - g(x_0) \right| \\ = \ & \left| \int K(x-y,t) \big(g(y) - g(x_0)\big) dy \right| \end{aligned}$$

As g is continuous, for $\varepsilon > 0$, there is some δ_0 such that

$$|g(y) - g(x_0)| < \frac{\varepsilon}{2}, \quad \text{if} \quad |y - x_0| < \delta_0,$$

so

$$\begin{aligned}
\left| \int_{|y-x_0|<\delta_0} K(x-y,t)\big(g(y) - g(x_0)\big)dy \right| &< \frac{\varepsilon}{2} \int_{|y-x_0|<\delta_0} K(x-y,t)dy \\
&< \frac{\varepsilon}{2} \int_{-\infty}^{\infty} K(x-y,t)dy \\
&= \frac{\varepsilon}{2}.
\end{aligned}$$

On the other hand, letting $M = \sup |g(x)|$, by Fact (c) there exists a small $\delta_1 > 0$ such that $\forall t \in [0, t_0]$,

$$\int_{|y-x|\geq\delta_0/2} K(x-y,t)dy < \frac{\varepsilon}{4M}.$$

For x satisfying $|x - x_0| < \delta_0/2$, we have, by the triangle inequality, $|y - x| \geq |y - x_0| - |x_0 - x| \geq \delta_0/2$ for y satisfying $|y - x_0| \geq \delta_0$. Therefore,

$$\begin{aligned}
\left| \int_{|y-x_0|\geq\delta_0} K(x-y,t)\big(g(y) - g(x_0)\big)dy \right| &\leq 2M \int_{|y-x_0|\geq\delta_0} K(x-y,t)dy \\
&\leq 2M \int_{|y-x|\geq\delta_0/2} K(x-y,t)dy \\
&= 2M \int_{|z|\geq\delta_1} K(z,t)dz \\
&< \frac{\varepsilon}{2}.
\end{aligned}$$

It follows that for any $\varepsilon > 0$, there exist δ_1 and $\delta_2 = \delta_0/2$ such that

$$|U(x,t) - g(x_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall x, t, \ |x - x_0| < \delta_2 = \delta_0/2, \ 0 \leq t < \delta_1.$$

□

A remarkable property of the heat equation is its smoothing property.

Proposition 9.5 *Assuming that g is bounded and continuous in $\mathbb{R}$, the solution given by (3.2.2) is smooth in $([a,b] \times [t,\infty))$ for any $t > 0$.*

Proof. This proof can be skipped. This follows from the boundedness of

$$\int \left| D_x^k K(x-y,t)g(y) \right| dy, \quad \int \left| D_t^k K(x-y,t)g(y) \right| dy,$$

for any order k. □

The next property is infinite speed of propagation. Consider a perturbation of the initial data g by some small function $h > 0$, the disturbance being very small, and vanishing outside the small interval $(y - \delta, y + \delta)$ where y is far away from x. Denote u the solution to the unperturbed problem and v the solution to the perturbed one. The solution has finite speed of propagation if v is equal to u at x for sufficiently small t. Otherwise it is infinite. It is easy to see that, for

$$v(x,t) = \int K(x-y,t)(g+h)dy,$$

we have

$$v(x,t) - u(x,t) = \int K(x-y,t)h(y)dy = \int_a^b K(x-y,t)h(y)dy$$

By Fact (b) $v(x,t) \neq u(x,t)$ for all time. This shows that the solution of the heat equation propagates at infinite speed. Certainly this is unrealistic. A spark on the moon can't be detected instantly on Earth. Our model is simply an approximate one. More realistic models are available. However, all of them are nonlinear equations.

It is clear that the solution decays to 0 as time goes to infinity. Since the energy keeps dissipating and there is no boundary condition, the temperature should tend to zero as $t \to \infty$. In fact, using

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int e^{-\frac{(x-y)^2}{4t}} g(y)dy,$$

we have

$$|u(x,t)| \leq \frac{M}{\sqrt{4\pi t}} \to 0, \text{ as } t \to \infty \quad \text{provided} \quad g \text{ is bounded.}$$